PRODUCTION AND OPERATIONS MANAGEMENT
2023/2024

Lisbon School of Economics & Management

ULISBOA UNIVERSIDADE DE LISBOA

Waiting-Line Models
Module D

Exercises and Resolutions – (1, 4), (2, 8, 10), (7, 13) and (5, 6, MC1, 9)

Class of 2021-04-28 – Exercises 1 and 4

Class of 2021-04-28 – Exercises 2, 8 and 10

Class of 2021-05-05 – Exercises 7 and 13

Class of 2021-05-05 – Exercises 5, 6, MC1 and 9

WL - Exer. 1

During busy periods, a new customer walks into StyleHair salon every 20 minutes. A typical service time at StyleHair salon lasts 12 minutes. Assume that service time follows a negative exponential distribution, and arrivals follow a Poisson distribution. Currently, the salon has one hairdresser.

Data:

$1/\lambda$ = 20 minutes $\Rightarrow$ λ = 3 clients/hour, Poisson arrivals

$1/\mu$= 12 minutes $\Rightarrow$ μ = 5 clients/hour, Exponential service

$\Rightarrow$ M/M/1 model

a) What is the average time a customer has to wait to get a haircut, $\mathbf{W_q}$?

$$W_q = \frac{\lambda}{\mu\,(\mu-\lambda)} = \frac{3}{5\times 2} = 0.3 \text{ hours} = \mathbf{18\ minutes}$$

b) What is the average time a customer is in the StyleHair salon, $\mathbf{W_s}$?

$W_s = W_q + 1/\mu = 18 + 12 =$ **30 minutes**

WL - Exer. 1

Data:

$1/\lambda$ = 20 minutes $\Rightarrow \lambda$ = 3 clients/hour, Poisson arrivals

$1/\mu$= 12 minutes $\Rightarrow \mu$ = 5 clients/hour, Exponential service

$\Rightarrow$ M/M/1 model

Little's Law

c) What is the average number of customers waiting to get a haircut, $\mathbf{L_q}$?

$$L_q = \frac{\lambda^2}{\mu\,(\mu-\lambda)} = \frac{9}{5(5-3)} = 0.9 \text{ clients, or } \mathbf{L_q = \lambda \times W_q} = 3 \times (18/60) = \mathbf{0.9\ clients}$$

d) What is the average number of customers in the StyleHair salon, $\mathbf{L_s}$?

$$L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{3}{2} = \mathbf{1.5\ clients}$$

WL - Exer. 1

Data:

$1/\lambda$ = 20 minutes $\Rightarrow \lambda$ = 3 clients/hour, Poisson arrivals

$1/\mu$= 12 minutes $\Rightarrow \mu$ = 5 clients/hour, Exponential service

$\Rightarrow$ M/M/1 model

e) What is the probability that the hairdresser is busy, **ρ?**

Utilization rate = $\rho = \lambda/\mu = 3/5 = 0.6$, or **60%**

f) What is the probability of arriving to "StyleHair" and find 1 customer in the queue?

One costumer in the queue ➔ Probability of 2 elements in the system, **P_2?**

$$P_2 = P(n>1) - P(n>2) = \left(\frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^3 = 0.1440$$

$=$ **14.40%**

WL - Exer. 4

At a loading and unloading dock, trucks arrive at the rate of 3 per day. The operations of loading and unloading are performed by a team of two men who, on average, serve four trucks per day (8 hours). It is estimated that for every additional man on the team, up to a maximum of six men, an extra truck is served. The hourly cost of a truck stopped (and its driver) is €40,00. The hourly cost of a man in the operations of loading and unloading is €12,00. Should we maintain the team of two men or change it? Justify your answer.

Data:
λ = 3 trucks/day, Poisson arrivals
μ = 4 trucks/day (team of 2 men), Exponential service
Truck waiting cost = €40,00/hour; Labour cost = €12,00/worker/hour

Although the **team** is comprised of more than one man, as these work together to unload a truck each time, **the model is M/M/1**, thus:

$$W_s = \frac{1}{(\mu - \lambda)}$$

WL - Exer. 4

Data:

λ = 3 trucks/day, Poisson arrivals

μ = 4 trucks/day (team of 2 men), Exponential service

Truck waiting cost = €40,00/hour; Labour cost = €12,00/worker/hour

For example, for $\mu = 4 \Rightarrow W_s = \dfrac{1}{(4-3)} = 1$ day, and therefore:

Team	μ	Ws
2	4	1 day
3	5	0.5 days
4	6	0.3333 days
5	7	0.25 days
6	8	0.20 days

WL - Exer. 4

Data:

λ = 3 trucks/day, Poisson arrivals

μ = 4 trucks/day (team of 2 men), Exponential service

Truck waiting cost = €40,00/hour; Labour cost = €12,00/worker/hour

Team	μ	Ws	Cost of truck stopped/day	Cost of labour/day	Total Cost/day
2	4	1 day	(€40 × 8) × 3 × 1 = €960,00	2 × €12 × 8 = €192,00	€1152,00
3	5	0.5 days	(€40 × 8) × 3 × 0.5 = €480,00	3 × €12 × 8 = €288,00	€768,00
4	6	0.33 days	(€40 × 8) × 3 × 0.33(3) = €319,97	4 × €12 × 8 = €384,00	**€703,97**
5	7	0.25 days	(€40 × 8) × 3 × 0.25 = €240,00	5 × €12 × 8 = €480,00	€720,00

The team **should be increased to *4 workers*,** as this guarantees the lowest total cost.

WL - Exer. 2

Paul has been collecting data at the “Stars” Snack Bar. He has found that, between 13:00 and 14:00 (01:00 – 02:00 PM), students arrive at the Snack Bar at a rate of 25 per hour (Poisson distributed), and service time takes an average of 2 minutes (exponential distribution). There is only one server, who can work on only 1 order at a time.

Data:

λ = 25 clients/hour, Poisson arrivals

$1/\mu$= 2 minutes $\Rightarrow$ μ = 30 clients/hour, Exponential service

S = 1 server $\Rightarrow$ M/M/1 model

a) What is the probability that a client does not need to wait to be served, $\mathbf{P_0}$?

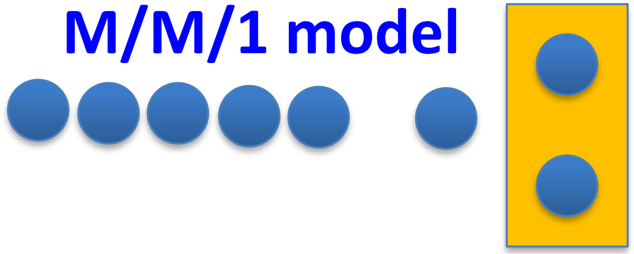
$P_0 = 1 - \rho = 1 - 25/30 = 0.1667$

b) What is the average number of clients in line, $\mathbf{L_q}$?

$$\mathbf{L_q} = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{25^2}{30 \times (30 - 25)} = 4.167 \text{ clients}$$

WL - Exer. 2

M/M/1 model



Data:
λ = 25 clients/hour, Poisson arrivals
$1/\mu$= 2 minutes $\Rightarrow$ μ = 30 clients/hour, Exponential service
S = 1 server $\Rightarrow$ M/M/1 model

c) What is the average time a client is in the Snack Bar area, W_s?

$W_s = W_q + 1/\mu = L_q/\lambda + 1/\mu = 4{,}167/25 + 2 = 10 + 2 =$ **12 minutes**

d) Suppose that a second server can be added to team up with the first (and, in effect act as one faster server). This will reduce the average service time to 90 seconds. How would this affect the average time a client is in the Snack Bar area?

Data: λ = 25 clients/hour, Poisson arrivals; μ = 3600/90 → 40 clients/hour, Exponential service

Ws $= 1/(\mu - \lambda) = 1/(40 - 25) = 0{,}0667$ hours = 4 minutes

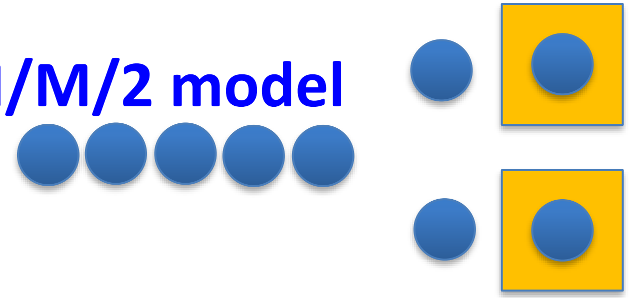
The time was **reduced by 8 minutes.**

WL - Exer. 2

e) Suppose that a second server is added and the two servers work independently, with each taking an average of 2 minutes. What would be the average time a student is in the system?

Data: λ = 25 clients/hour; μ = 30 clients/hour.
In this case, there are 2 independent servers, and thus the model is **M/M/2**.

M/M/2 model



$$P_0 = \frac{1}{\left(1 + \frac{25}{30}\right) + \frac{1}{2} \times \left(\frac{25}{30}\right)^2 \times \frac{2 \times 30}{2 \times 30 - 25}} = 0.4118$$

$$Wq = \frac{30 \times \left(\frac{25}{30}\right)^2}{(2-1)!\,(2 \times 30 - 25)^2} \times 0.4118 = 0.007 \text{ hours} = 0.42 \text{ min}$$

$W_s = W_q + 1/\mu = 0.42 + 2 =$ **2.42 minutes**

The time was **reduced by 9.58 minutes.**

WL - Exer. 8

The insurance company INSURANCE+ will open a new branch, ALMAINSURANCE+, in the iconic Rua das Flores in Almada. From experience, the operations manager observed service time is variable, approximately following a negative exponential distribution with an average of 20 minutes. A market research study further revealed that the arrival of clients proceeds at a rate of 5 per hour, according to a Poisson distribution. The operations manager will hire three employees to work independently, in two counters that serve a single waiting line. The daily wages will be of €30,00 per worker. The estimated hourly waiting cost is €1,50 per client.
The branch will be operating 8 hours per day.

Data:

λ = 5 clients/hour, Poisson arrivals; $1/\mu$= 20 minutes $\Rightarrow$ μ = 3 clients/hour, Service time follows a Negative Exponential Distribution; S = 3 servers $\Rightarrow$ M/M/3 model

WL - Exer. 8

Data:

λ = 5 clients/hour, Poisson arrivals; 1/μ= 20 minutes ⇒ μ = 3 clients/hour, Service time follows a Negative Exponential Distribution; S = 3 servers ⇒ M/M/3 model

a) What is the average time a client waits before being serviced, $\mathbf{W_q}$?

$$P_0 = \frac{1}{\left(1 + \frac{5}{3} + \frac{1}{2} \times \left(\frac{5}{3}\right)^2\right) + \frac{1}{3!} \times \left(\frac{5}{3}\right)^3 \times \frac{3 \times 3}{3 \times 3 - 5}} = \frac{1}{(4.0556 + 0{,}7716 \times 2{,}25)} = 0.1727$$

$$Wq = \frac{\mu \times \left(\frac{\lambda}{\mu}\right)^S}{(S-1)!\,(S\mu - \lambda)^2} P_0$$

$$\mathbf{W_q} = \frac{3 \times \left(\frac{5}{3}\right)^3}{(3-1)!\,(3 \times 3 - 5)^2} \times 0.1727 = \frac{13{,}8889}{32} \times 0.1727 = 0.0750 \text{ hours} = \mathbf{4.4974\ minutes}$$

WL - Exer. 8

Data:

λ = 5 clients/hour, Poisson arrivals; $1/\mu$= 20 minutes $\Rightarrow$ μ = 3 clients/hour, Service time follows a Negative Exponential Distribution; S = 3 servers $\Rightarrow$ M/M/3 model

b) How many hours per week an employee expects to spend with customers?

$$\rho = \frac{\lambda}{S\mu} = \frac{5}{3 \times 3} = 0.5556$$

Time spent with customers = $5 \times 8h \times 0.5556$ = **22.222 hours per week**

c) What is the total daily cost of the queuing system implemented at the ALMAINSURANCE+ branch?

Data: Waiting cost **(W_C)** = €1,5/hour; Labour cost **(L_C)** = €30,00/day

Total cost = $\lambda \times 8h \times W_q \times W_C + 3 \times L_C = 5 \times 8 \times 0.0750 \times$ €1,50 + 3 × €30,00 = **€94,50/day**

WL - Exer. 8

Data:

λ = 5 clients/hour, Poisson arrivals; $1/\mu$= 20 minutes $\Rightarrow$ μ = 3 clients/hour, Service time follows a Negative Exponential Distribution; S = 3 servers $\Rightarrow$ M/M/3 model

d) If the arrival rate increases to 14 clients per hour, what is the minimum number of employees that the operations manager needs to hire?

λ **= 14 clients/hour**, arrivals follow a Poisson distribution

M/M/S model ➔ $S \times \mu > \lambda \Leftrightarrow S \times 3 > 14 \Leftrightarrow S > 4.67 \Rightarrow$ **S = 5**

WL - Exer. 10

The wheat harvesting season in the American Midwest is short, and farmers deliver their truckloads of wheat to a giant central storage bin within a 2-week span. Because of this, wheat-filled trucks waiting to unload and return the fields have been known to back up for a block at the receiving bin. The central bin is owned cooperatively, and it is to every farmer's benefit to make the unloading/storage process as efficient as possible. The cost of grain deterioration cause by unloading delays and the cost of truck rental and idle driver time are significant concerns to the cooperative members. Although farmers have difficulty quantifying crop damage, it is easy to assign a waiting unloading cost for truck and driver of \$18 per hour. During the 2-week harvest season, the storage bin is open and operated 16 hours per day, 7 days per week, and can unload 35 trucks per hour according to an exponential negative distribution. Full trucks arrive all day long (during the hours the bin is open) at a rate of about 30 per hour, following a Poisson pattern.

To help the cooperative get handle on the problem of lost time while trucks are waiting in line or unloading at the bin, find the following:

Data: λ = 30 trucks/hour, arrival follow a Poisson distribution; μ = 35 trucks/hour, service time follows a negative exponential distribution; Waiting cost for trucks (and drivers) = 18 Euros/truck/hour; Working hours = 16 hours per day (7 days a week); M/M/1 model

WL - Exer. 10

Data: λ = 30 trucks/hour, arrival follow a Poisson distribution; μ = 35 trucks/hour, service time follows a negative exponential distribution; Waiting cost for trucks (and drivers) = 18 Euros/truck/hour; Working hours = 16 hours per day (7 days a week); M/M/1 model

a) The average number of trucks in the unloading system, $\mathbf{L_s}$?

$$L_s = \frac{\lambda}{(\mu - \lambda)} = \frac{30}{35 - 30} = \boldsymbol{6\ trucks}$$

b) The average time per truck in the system, W_s?

$$W_s = \mathrm{L_s}/\lambda = 6/30 = 0.2 \text{ hours} = \mathbf{12\ minutes}$$

WL - Exer. 10

Data: λ = 30 trucks/hour, arrival follow a Poisson distribution; μ = 35 trucks/hour, service time follows a negative exponential distribution; Waiting cost for trucks (and drivers) = 18 Euros/truck/hour; Working hours = 16 hours per day (7 days a week); M/M/1 model

c) The utilization rate for the bin area, ρ?

Utilization rate = $\rho = \lambda / \mu$ = 30/35= 0.8571, or **85.71%**

d) The probability that there are three or more trucks in the system at any given time?

$P(n \geq 3)$ = $P(n > 2) = (30/35)^3$ = 0.6297, or **62.97%**

e) The total daily cost to the farmers of having their trucks tied up in the unloading process?

W_s = 0.2 hours

Unloading Cost = 30 trucks/hour × 0.2 hours × 16 hours/day × €18/hour = **€1 728,00/day**

WL - Exer. 10

Data: λ = 30 trucks/hour, arrival follow a Poisson distribution; μ = 35 trucks/hour, service time follows a negative exponential distribution; Waiting cost for trucks (and drivers) = 18 Euros/truck/hour; Working hours = 16 hours per day (7 days a week); M/M/1 model

f) As mentioned, the cooperative uses the storage bin heavily only two weeks per year. Farmers estimate that enlarging the bin would cut unloading costs by 50% next year. It will cost €9000,00 to do so during the off-season. Would it be worth the expense to enlarge the storage area?

Total cost 2 weeks (**without enlargement**) = €1 728,00/day × 14 days = **€24 192,00**

Total cost for two weeks (**with enlargement**) = €9 000,00 + (€24 192,00/2) = **€21 096,00**

Based on the total costs, we can conclude that ***it is worth enlarging the bin*.**

WL - Exer. 7

Relax is a company that offers massage and beauty services in several Airports. The company has just opened a new store, RELAXFARO, at Faro Airport. Every hour, on average 4 passengers arrive to RELAXFARO to receive a massage, according to a Poisson distribution. Currently, RELAXFARO has only one masseur, that takes 12 minutes to do a massage. After finishing the massage, about 50% of the passengers also use the beauty service. The beauty service has one employee that takes, on average, 20 minutes to execute the beauty service. The service time of the beauty service follows a negative exponential distribution.

Massage Service Data:

λ = 4 passengers/hour, Poisson arrivals; $1/\mu$ = 12 minutes $\Rightarrow$ μ = 5 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/D/1 model**

Beauty Service Data:

λ = 2 passengers/hour, Poisson arrivals; $1/\mu$ = 20 minutes $\Rightarrow$ μ = 3 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/M/1 model**

WL - Exer. 7

Massage Service Data:

λ = 4 passengers/hour, Poisson arrivals; $1/\mu$ = 12 minutes $\Rightarrow$ μ = 5 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/D/1 model**

Beauty Service Data:

λ = 2 passengers/hour, Poisson arrivals; $1/\mu$ = 20 minutes $\Rightarrow$ μ = 3 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/M/1 model**

a) On average, how long does a passenger wait in the massage service?

$$W_q = \frac{4}{2 \times \mu\,(\mu - \lambda)} = \frac{4}{2 \times 5\,(5 - 4)} = 0.4 \text{ hours} \Rightarrow \mathbf{24\ minutes}$$

b) What is the probability of, at any given moment, having 2 or more passengers waiting in the beauty service? **P(n≥3)**

$P(n > k) = \rho^{k+1}$ (probability of more than *k* units **in the system**, where *n* is the number of units **in the system**)

$P(n \geq 3) = P(n>2) = (2/3)^{2+1} =$ **0.2963**

WL - Exer. 7

Massage Service Data:

λ = 4 passengers/hour, Poisson arrivals; $1/\mu$ = 12 minutes $\Rightarrow$ μ = 5 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/D/1 model**

Beauty Service Data:

λ = 2 passengers/hour, Poisson arrivals; $1/\mu$ = 20 minutes $\Rightarrow$ μ = 3 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/M/1 model**

c) What is the probability that, at any given moment, there are four passengers in the beauty service? **P4?**

$P(n > k) = \rho^{k+1}$

$P(n > 3) - P(n > 4) = (2/3)^{3+1} - (2/3)^{4+1} = 0.0658$,

or

$P_n = P_0 \times (\lambda/\mu)^n$

$P_4 = (1-2/3) \times (2/3)^4 = 0.0658$

WL - Exer. 7

Massage Service Data:

λ = 4 passengers/hour, Poisson arrivals; $1/\mu$ = 12 minutes $\Rightarrow$ μ = 5 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/D/1 model**

Beauty Service Data:

λ = 2 passengers/hour, Poisson arrivals; $1/\mu$ = 20 minutes $\Rightarrow$ μ = 3 passengers/hour, Service time follows a Negative Exponential Distribution; S = 1 server $\Rightarrow$ **M/M/1 model**

d) On average, how many passengers are in RELAXFARO? **Ls_{Total} = $Ls_{Massage}$ + Ls_{Beauty}**

Massage Service: $W_q = 0.4 \text{ hours} \Rightarrow L_q = \lambda \times W_q = 4 \times 0.4 = 1.6 \text{ passengers} \Rightarrow L_S = L_q + \left(\frac{\lambda}{\mu}\right) = 1.6 + \left(\frac{4}{5}\right) = 2.4 \text{ passengers}$

Beauty Service: $L_S = \frac{\lambda}{(\mu- \quad)} = \frac{2}{(3-2)} = 2 \text{ passengers}$

Ls_{Total} = $Ls_{Massage}$ + Ls_{Beauty} ➔ Ls_{Total} = 2.4 + 2 **= 4.4 passengers**

WL - Exer. 13

MATUTIX is a chips factory fully automated. Currently, the factory has 5 computers that run the production process. These computers are monitored by only one technician. It takes an average of 15 minutes (exponentially distributed) to adjust a computer that develops a problem. Each computer develop a problem at a rate of 0.7059 per hour.

Data:

λ = 0.706 problems/computer/hour, Poisson arrivals; μ = 1 adjustment/15 minutes $\Rightarrow$

μ = 4 adjustments/hour, Service time follows a Negative Exponential Distribution;

N = 5; S = 1 server $\Rightarrow$ **M/M/1/5 model**

WL - Exer. 13

Data:

λ = 0.706 problems/computer/hour, Poisson arrivals; μ = 1 adjustment/15 minutes $\Rightarrow$
μ = 4 adjustments/hour, Service time follows a Negative Exponential Distribution;
N = 5; S = 1 server $\Rightarrow$ M/M/1/5 model

a) Determine the average number of computers waiting for adjustments. **L_q?**

$$P_0 = \frac{1}{\sum_{n=0}^{N} \frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{\sum_{n=0}^{5} \frac{5!}{(5-n)!}\left(\frac{.706}{4}\right)^n}$$

$$= \frac{1}{1(.1765)^0 + 5(.1765)^1 + 20(.1765)^2 + 60(.1765)^3 + 120(.1765)^4 + 120(.1765)^5}$$

$$= \frac{1}{1 + 0.8825 + 0.6230 + 0.3299 + 0.1165 + 0.0206} = \frac{1}{2.9725} = 0.3364$$

$$L_q = N - \left(\frac{\lambda+\mu}{\lambda}\right)(1 - P_0) = 5 - \left(\frac{.706+4}{.706}\right)(1 - .3364) = 5 - 4.423 = 0.577 \text{ computers}$$

L_q = N – [(λ + μ)/ λ] x (1 – P_0) = 5 – [(0.706 + 4)/0.706] x (1 – 0.3364) = **0.577 computers**

WL - Exer. 13

Data:

λ = 0.706 problems/computer/hour, Poisson arrivals; μ = 1 adjustment/15 minutes $\Rightarrow$
μ = 4 adjustments/hour, Service time follows a Negative Exponential Distribution;
N = 5; S = 1 server $\Rightarrow$ M/M/1/5 model

b) Determine the average number of computers being adjusted (or in the system). **L_s?**

$P_0 = 1 - (\bar{\lambda}/\mu) \Rightarrow 1 - P_0 = \bar{\lambda}/\mu \rightarrow \bar{\lambda}/\mu = 0.6636$

We know that $L_s = L_q + \bar{\lambda}/\mu$, then:

L_s $= L_q + \bar{\lambda}/\mu = 0{,}577 + 0{,}6636 =$ **1.241 computers**

WL - Exer. 5

At the LOGIST Company there are 4 employees responsible for receiving the orders that arrive in the trucks. The LOGIST Director is planning to change the number of employees in order to reduce some costs. The trucks arrive to the LOGIST according to a Poisson rate of 1 per hour. The necessary average time to receive the orders that arrive in the trucks follows an exponential distribution that depends on the number of employees doing the job (suppose that 1 μ = 1 n , where n is the number of employees). The cost of each employee is €20,00 per hour. The trucks' waiting time is worth €0,50 per minute.

Data:

λ = 1 truck/day, Poisson arrivals; Service follows a Exponential Distribution; **M/M/1 model**

Labor costs: €20,00/hour/man; Trucks' waiting time costs: €0,50/minute ➔ €30,00/hour

WL - Exer. 5

a) What is the optimal number of employees?

n	1/μ = 1/n	μ	Wq	Total cost/hour
1	1 hour	1 truck/hour	----	----
2	0.5 hours	2 trucks/hour	**0.5 hours**	2×€20+€30×1×0.5 = 55,00 Euros
3	1/3 hour	3 trucks/hour	**1/6 hours**	3×€20+€30×1×(1/6) = 65,00 Euros
4	¼ hour	4 trucks/hour	**1/12 hours**	4×€20+€30×1×(1/12) = 82,50 Euros

$$W_q = \frac{\lambda}{\mu\,(\mu-\lambda)} = \frac{1}{2(2-1)} = \mathbf{0.5\ hours};\ W_q = \frac{1}{3(3-1)} = \mathbf{0.166\ hours};\ W_q = \frac{1}{4(4-1)} = \mathbf{0.0833\ hours};$$

The optimal number of employees is 2.

WL - Exer. 5

b) After unloading all the orders, each truck is immediately washed by the employees. The necessary time to wash the trucks follows an exponential distribution with an average of 20 minutes. What is the average time that each truck spends in the LOGIST? (Consider that the optimal number of employees is 2 and they always work together).

Total time = $1/\mu 1 + 1/\mu 2$ = 30 min + 20 min = 50 minutes

μ = 60/50 = 1.2 trucks/hour; λ = 1 truck/hour

M/M/1 model

$$W_s = \frac{\lambda}{(\mu - \lambda)} = \frac{1}{(1.2 - 1)} = \textbf{5 hours}$$

WL - Exer. 6

At a hospital emergency room, patients arrive at the rate of 8 per hour. The Manager of the hospital wants that a patient does not wait for the doctor, on average, more than 10 minutes.

Data:

λ = 8 patients/hour; Assuming that the arrivals follows a Poisson distribution, and that the service time follows a negative Exponential distribution ➔ **M/M/1 model**

$W_q \leq 10$ minutes

WL - Exer. 6

Data: λ = 8 patients/hour; Assuming that the arrivals follows a Poisson distribution, and that the service time follows a negative Exponential distribution ➔ **M/M/1 model; $W_q \leq$ 10 minutes**

a) Determine the average service time of medical staff per patient ($\frac{1}{\mu}$), and the other measures of system performance.

$1/\mu$ = ?

$$W_q = \frac{\lambda}{\mu\,(\mu-\lambda)} \leq \frac{10}{60} \Rightarrow \frac{8}{\mu\,(\mu-8)} \leq \frac{10}{60} \Rightarrow \mu^2 - 8\mu - 6\times 8 \geq 0 \Rightarrow$$

$$\boldsymbol{\mu = 12} \text{ or } \mu = -4$$

Therefore the service time of the team of doctors is 1/12 hour = **5 minutes**

WL - Exer. 6

Data: λ = 8 patients/hour; Assuming that the arrivals follows a Poisson distribution, and that the service time follows a negative Exponential distribution ➔ **M/M/1 model; $W_q \leq$ 10 minutes**

a) Determine the average service time of medical staff per patient ($\frac{1}{\mu}$), and the other measures of system performance.

W_S = 10 + 5 = **15 minutes** (W_S = Waiting time + Service time)

$$L_s = \lambda \times W_S = 8 \times \left(\frac{15}{60}\right) = \mathbf{2\ patients}$$

$L_q = W_q \times \lambda$ = (10/60) × 8 = **1.333 patients**

$\rho = (\lambda/\mu)$ = 8/12 = 0.667 ➔ **66.67%**; $P_0 = 1 - \rho$ = 0.333 ➔ **33.33%**

WL - Exer. 6

Data: λ = 8 patients/hour; Assuming that the arrivals follows a Poisson distribution, and that the service time follows a negative Exponential distribution ➔ **M/M/1 model; $W_q \leq$ 10 minutes**

b) The manager decided that if the expected waiting time for a patient exceed one hour it will be transferred to the nearest support hospital. What is the expected number of patients to be transferred per year? (assume 365 days per year and an emergency operation of 24 hours per day).

If W_q > 60 minutes $\Rightarrow$ the patient is transferred

5 patients in the queue (6 in the system)

$$P(n > 5) = \left(\frac{\lambda}{\mu}\right)^6 = \left(\frac{8}{12}\right)^6 = 0.087791$$

$(8\times24h\times365) \times 0.087791 = \mathbf{6.152\ patients/year}$

6 patients in the queue (7 in the system, the seventh has to be transferred and abandons the system)

$$P(n > 6) = \left(\frac{\lambda}{\mu}\right)^7 = \left(\frac{8}{12}\right)^7 = 0.058528$$

$(8\times24h\times365) \times 0.058528 = \mathbf{4.102\ patients/year}$

WL – MChoice 1

CANNEDHEAT produces canned tuna using 25 canning machines. On average, each machine breaks-down 2 times per hour, following a Poisson process. CANNEDHEAT employs 5 maintenance technicians. The repair time follows a negative exponential distribution, with an average of 6 minutes. The utilization rate of the technicians (**ρ**) is 80%, and P_0 = 0.0085.

Data:

λ = 2 breaks-down/machine/hour, Poisson arrivals; **μ** = 10 breaks-down/hour, Service time follows a Negative Exponential Distribution; **N** = 25; **S** = 5 ⇒ **M/M/5/25 model**

WL – MChoice 1

Data:

λ = 2 breaks-down/machine/hour, Poisson arrivals; μ = 10 breaks-down/hour, Service time follows a Negative Exponential Distribution; **N** = 25; **S** = 5 $\Rightarrow$ **M/M/5/25 model**

What is the probability of all 4 machines being broken down at the same time? **P_4?**

$$P_n = \begin{cases} \dfrac{N!}{(N-n)!\,n!} \times \left(\dfrac{\lambda}{\mu}\right)^n \times P_0, & \text{if } n = 1, \ldots\ldots S \\ \dfrac{N!}{(N-n)!\,S!\,S^{n-S}} \times \left(\dfrac{\lambda}{\mu}\right)^n \times P_0, & \text{if } n = S, \ldots\ldots N \\ 0, & \text{if } n > N \end{cases}$$

When n < S

$P_4 = 25!/[(25 - 4)! \times 4!] \times (2/10)^4 \times 0.0085 =$ **0.1720**

What is the probability of all 4 machines being broken down at the same time?		
1	X	0.172
2		0.20
3		0.80
4		0.828

WL – MChoice 1

Data:

λ = 2 breaks-down/machine/hour, Poisson arrivals; μ = 10 breaks-down/hour, Service time follows a Negative Exponential Distribution; **N** = 25; **S** = 5 $\Rightarrow$ **M/M/5/25 model**

What is the average time a machine has to wait to be repaired? **W_q?**

$\rho = \bar{\lambda} / S\mu \rightarrow 0.8 = \bar{\lambda}/(5 \times 10) \rightarrow \bar{\lambda} = 40$ breaks-down/hour

$$\bar{\lambda} = \sum_{n=0}^{N-1} (N-n)\lambda P_n = \lambda(N-L)$$

$\bar{\lambda} = \lambda \times (N - L_s) \rightarrow 40 = 2 \times (25 - L_s) \rightarrow L_s = 5$

$L_s = \bar{\lambda} \times W_s \rightarrow W_s = 5/40 = 0.1250$ hours

W_q $= W_s - (1/\mu) \rightarrow W_q = 0.1250$ hours $- (1/10) =$ 0.0250 hours = **1.5 minutes**

What is the average time a machine has to wait to be repaired?		
1		7.5 minutes
2		6 minutes
3	X	**1.5 minutes**
4		1.03 minutes

WL - Exer. 9

The manager of the Smile Hospital faces the problem of providing treatment for patients who arrive at different rates during the day. There are only 4 doctors available to treat patients when needed. If not needed, they can be assigned other responsibilities or rescheduled to work at other hours. Patients are treated on a first-come, first-served basis and see the first available doctor. Treatment times follow the exponential pattern and takes, on average, 12 minutes. The arrival pattern for a typical day follows a Poisson distribution and is as follows:

Time	Arrival Rate
9 am – 3 pm (09:00 – 15:00)	6 patients/hour
3 pm – 8 pm (15:00 – 20:00)	4 patients/hour
8 pm – midnight (20:00 – 24:00)	12 patients/hour

Management feels that, on average, patients should not have to sit in the waiting area for more than 5 minutes before being seen by a doctor. It is known that for the third period indicated the probability of 0 units in the system is 5/89 when S=3 and 25/301 when S=4.

WL - Exer. 9

Data: 4 doctors (maximum); Arrivals follows a Poisson distribution according to the table; $1/\mu = 12$ minutes ➔ $\mu = 5$ patients/hour, and the service time follows a negative Exponential distribution;
Time in the Waiting Room is less than 5 minutes ➔ **$W_q \leq 5$ minutes**
From 20:00 to 24:00 for S=3, $P_0=5/89$, and for S=4, $P_0=25/301$

Time	Arrival Rate
9 am – 3 pm (09:00 – 15:00)	6 patients/hour
3 pm – 8 pm (15:00 – 20:00)	4 patients/hour
8 pm – midnight (20:00 – 24:00)	12 patients/hour

How many doctors should be on duty during each period to maintain the level of patient care expected?

WL - Exer. 9

From 09:00 to 15:00	λ = 6 patients/hour μ = 5 patients/hour

$S\mu > \lambda \Leftrightarrow S > 6/5 \Rightarrow$ **S=2 ⇒ M/M/2**

$$P_0 = \frac{1}{\left(1+\frac{6}{5}\right)+\frac{1}{2!}\times\left(\frac{6}{5}\right)^2\times\frac{2\times 5}{2\times 5-6}} = 0.25$$

$$W_q = \frac{5\times\left(\frac{6}{5}\right)^2}{(2-1)!\,(2\times 5-6)^2}\times 0.25 = 0.1125 \text{ hours} = 6.75 \text{ minutes} > 5 \text{ minutes}$$

S=3 ⇒ M/M/3; $P_0 = 0.2941$

$$W_q = \frac{5\times\left(\frac{6}{5}\right)^3}{(3-1)!(3\times 5-6)^2}\times 0.2941 = 0.01568 \text{ hours} = 0.9411 \text{ minutes} < 5 \text{ minutes}$$

WL - Exer. 9

From 15:00 to 20:00	λ = 4 patients/hour μ = 5 patients/hour

$S\mu > \lambda \Leftrightarrow S > 4/5 \Rightarrow$ **S=1 $\Rightarrow$ M/M/1**

$$\mathbf{W_q} = \frac{\lambda}{\mu\,(\mu - \lambda)} = \frac{4}{5 \times (1)} = 0.8 \text{ hours} = \mathbf{48\ minutes} > \mathbf{5\ minutes}$$

S=2 $\Rightarrow$ M/M/2; $P_0 = 0.4286$

$$P_0 = \frac{1}{\left(1 + \frac{4}{5}\right) + \frac{1}{2!} \times \left(\frac{4}{5}\right)^2 \times \frac{2 \times 5}{2 \times 5 - 4}} = 0.4286$$

$$\mathbf{W_q} = \frac{5 \times \left(\frac{4}{5}\right)^2}{(2-1)!(2\times5-4)^2} \times 0.4286 = 0.038 \text{ hours} = \mathbf{2.28\ minutes} < \mathbf{5\ minutes}$$

WL - Exer. 9

From 20:00 to 24:00	λ = 12 patients/hour μ = 5 patients/hour

$S\mu > \lambda \Leftrightarrow S > 12/5 \Rightarrow$ **S=3 ⇒ M/M/3** $P_0 = 5/89$

$$W_q = \frac{5 \times \left(\frac{12}{5}\right)^3}{(3-1)!\,(3 \times 5 - 12)^2} \times \frac{5}{89} = 0.2157 \text{ hours} = \mathbf{12.9438 \text{ minutes}}$$ **> 5 minutes**

S=4 ⇒ M/M/4; $P_0 = 25/301$

$$W_q = \frac{5\times\left(\frac{12}{5}\right)^4}{(4-1)!(4\times5-12)^2} \times \frac{25}{301} = 0.0359 \text{ hours} = \mathbf{2.1528 \text{ minutes}}$$ **< 5 minutes**

WL - Exer. 9

How many doctors should be on duty during each period to maintain the level of patient care expected?

Hour	λ	Number of doctors on duty
From 09:00 to 15:00	6 patients	3
From 15:00 to 20:00	4 patients	2
From 20:00 to 24:00	12 patients	4

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